Topological properties and classes of functions defined using neighbourhood assignments

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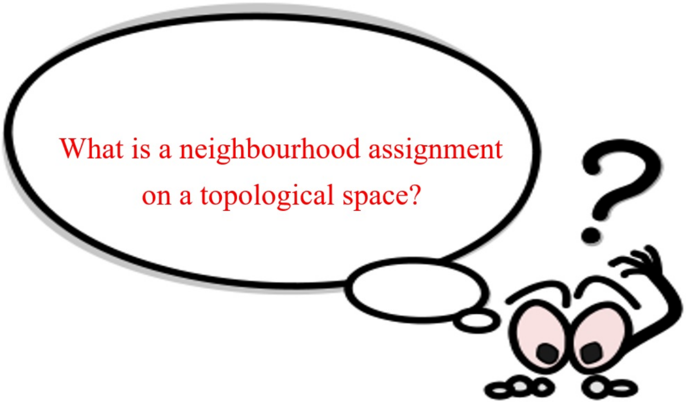
Abstract

In this talk, we will make a survey of the recent work on the characterization of some of the topological properties using neighbourhood assignments. We also present some classes of functions which are defined using neighbourhood assignments.

Neighbourhood assignment



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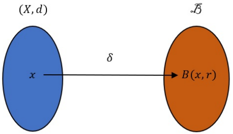

Example

Let $X = \mathbb{R}$ be equipped with the Euclidean metric and $r > 0$. The mapping define below is a neighbourhood assignment on X.

$$\delta(x) = (x - r, x + r), \qquad x \in X.$$

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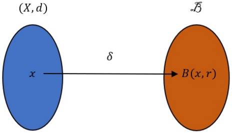

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Let (X, τ) be a topological space. **A neighbourhood assignment** on (X, τ) is a function $\delta : X \to \tau$ such that $x \in \delta(x)$ for each $x \in X$.

Denote the collection of all neighbourhood assignments on X by $\Delta(X)$.

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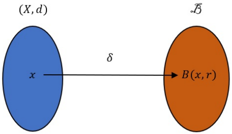

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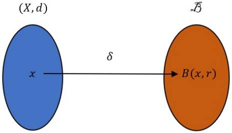

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Example

Let $X = \mathbb{R}$ be equipped with the co-countable topology $\tau_{co-count}$.Define an assignment $\delta : X \to \tau_{co-count}$ by letting $\delta(x) = (\mathbb{R} - \mathbb{N}) \cup \{x\}$ for all $x \in X$. Then $\delta \in \Delta(X)$.

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(1) *D*-Space

Definition

A space X is called compact if and only if for any $\delta \in \Delta(X)$ there is a finite subset $Y \subseteq X$ such that

$$\bigcup\{\delta(x) : x \in Y\} = X.$$

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If we substitute "closed discrete" for "finite" then we obtain the definition of the class of *D*-spaces (Van Douwen-1977).

Properties of D-space

D-Space but not Compact

Standard Euclidean space $\mathbb{R}$ is a D-space but not compact.

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Theorem

1. *Every T_1 compact space/σ-compact space is a D-space.*
2. *Every countably compact D-space is compact.*

(2) Gauge Compact space (Zhao-2005)

Definition

A topological space X is called gauge compact *if for any $\delta \in \Delta(X)$, there is a finite set $A \subseteq X$ such that for any $x \in X$ there is $a \in A$ such that $x \in \delta(a)$ or $a \in \delta(x)$.*

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Example Gauge compact

Let $X = \mathbb{N}$ be the set of all positive integers. A set U is open in X if and only if $U = \emptyset$ or U contains 1.
Obviously, X is not compact.
Now, let $\delta \in \Delta(X)$. Then every $x \in X$, $1 \in \delta(x)$. Thus, X is gauge compact.

(3) Gauge Compact Index

Definition

Let X be a topological space. The gauge compact index of X, denoted by $GCI(X)$, is defined as

$$GCI(X) = \inf\{\beta : \forall\delta \in \Delta(X), \exists A \subseteq X \textit{ so that } |A| < \beta \wedge X \prec^M_\delta A\},$$

where β is a cardinal number and $|A|$ is the cardinality of set A.

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Example

Let $X = \mathbb{N}$ be the set of all positive integers. A set U is open in X if and only if $U = \emptyset$ or U contains 1. Since every $x \in X$, $1 \in \delta(x)$. Thus, $GCI(X) = 2$.

Properties

Properties of gauge compact

1. Every Tychonoff gauge compact space is compact.
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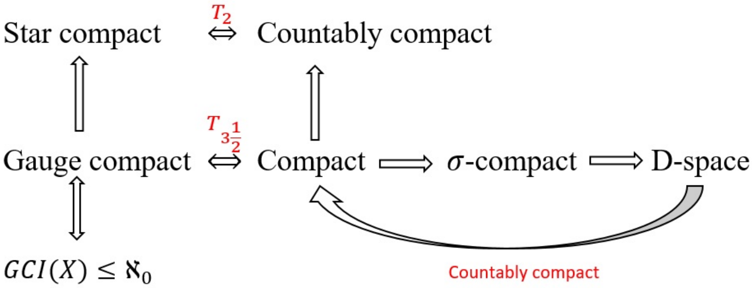
Properties of GCI

1. Let $n > 1$ and X be a Hausdorff space. Then $GCI(X) = n$ iff $|X| = n - 1$.
2. For any T_1 space X, $GCI(X) = 2$ iff $|X| = 1$.
4. Let X be a T_1 space with $|X| > 2$. If $GCI(X) = 3$ then X is hyperconnected.

Relation between gauge compact with other compactness



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Definition

1. A function $f : X \to Y$ between two topological spaces is of Baire class one *if there is a sequence $\{f_n\}$ of continuous functions $f_n : X \to Y$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ holds for every x.*

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2. A function $f : X \to Z$ from a topological space X to a metric space (Z, d) is weakly separated *if for any $\varepsilon > 0$, there is a $\delta \in \Delta(X)$ such that*

$$d(f(x), f(y)) < \varepsilon \text{ whenever } (x, y) \in \delta(y) \times \delta(x).$$

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3. A function $f : X \to Y$ between two topological spaces has the point of continuity property (PCP) *if the restriction of f to any non-empty closed set of X has a point of continuity. (A. Bouziad-2012)*

Properties (Lee, Tang, Zhao-2001)

Theorem

Let $f : X \to Y$ be a function from two Polish spaces. Then the following are equivalent:
(1) f is Baire class one.
(2) f is weakly separated.
(3) f has PCP.

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Question (A. Bouziad-2012)

Does every weakly separated function $f : X \Rightarrow (Y, d)$ from a compact space X to a metric space (Y, d) have a point of continuity?

Example

Let $X = (N, \tau_{cof})$ be the set of all natural numbers with the co-finite topology τ_{cof}.
(1) X is compact.
(2) Let $f : X \Rightarrow R$ be the function such that

$$f(2k) = 0 \text{ and } f(2k-1) = 1, \quad k = 1, 2, \cdots .$$

Then f does not have a point of continuity.
But f is weakly separated: Let $\delta \in \Delta(X)$ be defined by

$$\delta(1) = X, \delta(n) = X - \{1, ..., n-1\}$$

Then for any $\epsilon > 0$, $(m, n) \in \delta(n) \times \delta(m)$ implies $m = n$, so trivially $d(f(m), f(n)) = 0 < \epsilon$.

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