Kurzweil-Stieltjes integral and its applications

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1. NOTATIONS

- $-\infty < a < b < \infty$,
- function $f\colon [a,b] \to R$ is *regulated* on $[a,b]$, if
  $f(s+) := \lim_{\tau\to s+} f(\tau) \in \mathbb{R}$ for $s \in [a,b)$, $f(t-) := \lim_{\tau \to t-} f(\tau) \in \mathbb{R}$ for $t \in (a,b]$.
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $G[a,b]$ (resp. G) is the space of regulated functions on $[a,b]$.
  (G is Banach space with respect to the norm $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$).
- $BV = BV[a,b] = \left\{ f\colon [a,b] \to \mathbb{R} : \operatorname{var}_a^b f < \infty \right\}$ is the space of functions with bounded variation.
- function $f\colon [a,b] \to R$ is *finite step function*, if there is a division
  $a = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = b$ of $[a,b]$ such that f is constant on every (α_{j-1}, α_j),
  $S[a,b]$ (or S) is the set of finite step functions on $[a,b]$.
- Regulated functions are uniform limits of finite step functions,
  they have at most countably many points of discontinuity.
  Every function f of bounded variation is a difference $f = g - h$
  of nondecreasing functions g and h.
- $S[a,b] \subset BV[a,b] \subset G[a,b]$.

2. DEFINITION OF KS INTEGRAL

Notation

- Positive functions $\delta\colon [a,b] \to (0,\infty)$ are gauges on $[a,b]$.
- Couples $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of ordered finite sets are partitions of $[a,b]$ if $\boldsymbol{\alpha} = \{\alpha_0 < \alpha_1 < \ldots < \alpha_{\nu(P)} = b\}$ is a division of $[a,b]$ and $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_{\nu(P)}\}$ are its tags, i.e. $\xi_j \in [\alpha_{j-1}, \alpha_j]$ for all j.
- $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is δ-fine if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ for all j.
- For $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$, $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ we set
$$S(f, dg, P) = \sum_{j=1}^{\nu(P)} f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})]\,.$$

Definition

$$I = \int_a^b f\,dg \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there is a gauge } \delta \text{ on } [a,b] \text{ such that} \\ \qquad\qquad \Big| S(f, dg, P) - I \Big| < \varepsilon \\ \text{for every } \delta - \text{fine partition } P. \end{array} \right.$$

$$\int_c^c f\,dg = 0, \quad \int_b^a f\,dg = -\int_a^b f\,dg.$$

- KS integral has usual linear properties and is an additive function of intervals.

- $\displaystyle\int_a^b f\, dg \in \mathbb{R} \implies \left|\int_a^b f\, dg\right| \leq \|f\|_\infty \, (\mathrm{var}_a^b\, g)\,, \quad \left|\int_a^b f\, dg\right| \leq 2\, \|f\|_{BV}\, \|g\|_\infty.$

- RS $\subset$ KS.

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- RS $\subset$ KS.

- $f \in G[a,b]$, $g \in G[a,b] \implies$

  Both integrals $\displaystyle\int_a^b f\,dg$ and $\displaystyle\int_a^b g\,df$ exist if one of the functions f, g is a finite step function.

3. FINITE STEP FUNCTIONS

Integration of finite step functions

- $f(x) \equiv c$, $g : [a,b] \to \mathbb{R} \implies \int_a^b f \; dg = c\,[g(b) - g(a)]$.
- $f : [a,b] \to \mathbb{R}$, $g(x) \equiv c \implies \int_a^b f \; dg = 0$.
- $g : [a,b] \to \mathbb{R}$ regulated, $\tau \in [a,b]$ and $f = \chi_{[\tau,b]} \implies \int_\tau^b f \; dg = g(b) - g(\tau)$.

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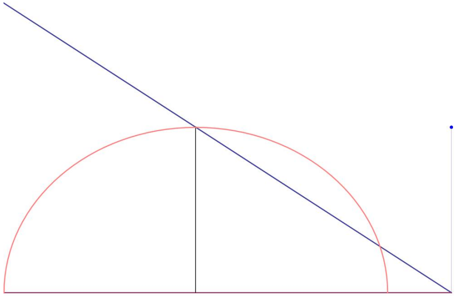
  Let $\delta(x) = \begin{cases} \frac{1}{4}(\tau - x) & \text{for } x < \tau\,, \\ \eta & \text{for } x = \tau \end{cases}$

  and let $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ be δ-fine. Then



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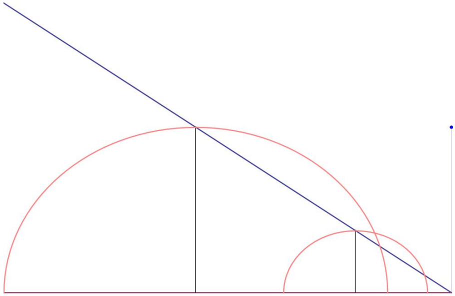

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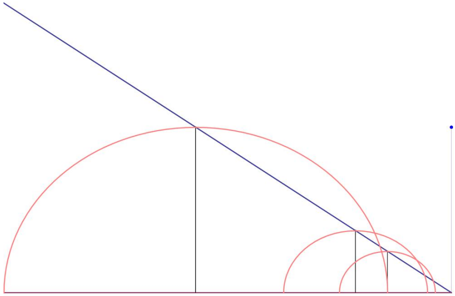

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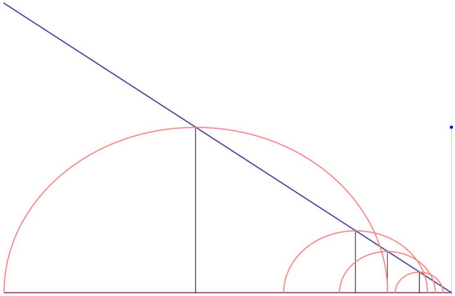

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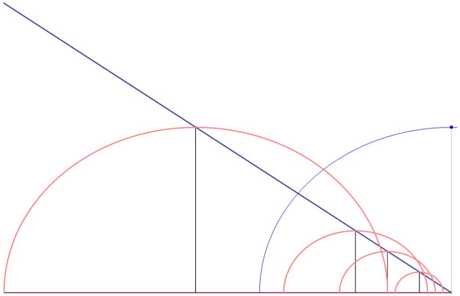

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  $\implies S(P) = [g(\tau) - g(\alpha_{m-1})] \to [g(\tau) - g(\tau-)]$

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$$\int_a^b \chi_{[\tau]}\, dg = \begin{cases} g(b) - g(b-) & \text{for } \tau = b\,, \\ g(\tau+) - g(\tau-) & \text{for } \tau \in (a,b)\,, \\ g(b) - g(b-) & \text{for } \tau = b\,, \end{cases}$$

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- $f : [a,b] \to \mathbb{R} \quad \tau \in [a,b] \implies$

$$\int_a^b f\, d\chi_{[a,\tau]} = \int_a^b f\, d\chi_{[a,\tau)} = -f(\tau), \quad \int_a^b f\, d\chi_{[\tau,b]} = \int_a^b f\, d\chi_{(\tau,b]} = f(\tau),$$

$$\int_a^b f\, d\chi_{[\tau]} = \begin{cases} -f(a) & \text{for } \tau = a\,, \\ 0 & \text{for } \tau \in (a,b)\,, \\ f(b) & \text{for } \tau = b\,. \end{cases}$$

4. EXISTENCE OF KS INTEGRAL

- $f \in G[a,b]$, $g \in G[a,b]$ $\implies \displaystyle\int_a^b f\, dg \in \mathbb{R}$ and $\displaystyle\int_a^b g\, df \in \mathbb{R}$
  if at least one of the functions f, g is a finite step function.

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- If
  - $f,\ f_k \in G[a,b],\quad g \in BV[a,b]$ for $k \in \mathbb{N}$,
  - $f_k \rightrightarrows f$,

  then $\displaystyle\int_a^b f_k\ dg \to \int_a^b f\ dg.$

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- $f \in BV[a,b],\ g \in G[a,b] \implies \int_a^b f\, dg \in \mathbb{R}$.

Let $\varepsilon > 0$ be given.

Choose finite step functions g_k in such a way that $g_k \rightrightarrows g$ on $[a,b]$.

Let $\|g_k - g_\ell\|_\infty < \varepsilon$ for $k, \ell \geq k_0$.

Then

$$\left| \int_a^b f \, d\,[g_k - g_\ell] \right| \leq 2\,\|g_k - g_\ell\|_\infty \, \|f\|_{BV} \leq 4\,\varepsilon\,\|f\|_{BV} \quad \text{for } k, \ell \geq k_0,$$

i.e. $\left\{ \int_a^b f \, d\,g_k \right\}$ is Cauchy.

Hence $\lim_{k \to \infty} \int_a^b f \, d\,g_k = I \in \mathbb{R}$.

Choose $K \geq k_0$ and a gauge δ on $[a,b]$ in such a way that

$$\left| \int_a^b f \, d\,g_K - I \right| < \varepsilon \quad \text{and} \quad \left| S(f, dg_K, P) - \int_a^b f \, d\,g_K \right| < \varepsilon \quad \text{for every } \delta\text{-fine } P\,.$$

Then

$$\left| S(f, dg, P) - I \right| \leq \left| S(f, dg, P) - S(f, dg_K, P) \right| + \left| S(f, dg_K, P) - \int_a^b f \, d\,g_K \right| + \left| \int_a^b f \, d\,g_K - I \right| < 2\varepsilon\,(\|f\|_{BV} + 1)$$

for every δ-fine P.

Theorem

<u>ASSUME</u>: $f \in G[a,b]$, $g \in G[a,b]$ and at least one of the functions f, g has a bounded variation.

<u>THEN</u>: both integrals $\displaystyle\int_a^b f\, dg$ and $\displaystyle\int_a^b g\, df$ exist.

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<u>THEN</u>: both integrals $\int_a^b f\,dg$ and $\int_a^b g\,df$ exist.

- KS = PS.
- $(\mathrm{LS})\int_{[c,d]} f\,dg \in \mathbb{R} \implies$

$$\int_c^d f\,dg \in \mathbb{R} \quad \text{and} \quad (\mathrm{LS})\int_{[c,d]} f\,dg = f(c)\,\Delta^- g(c) + \int_c^d f\,dg + f(d)\,\Delta^+ g(d).$$

Theorem

ASSUME: $f \in G[a,b]$, $g \in G[a,b]$ and at least one of the functions f, g has a bounded variation.

THEN: both integrals $\int_a^b f\,dg$ and $\int_a^b g\,df$ exist.

- KS = PS.
- $$(\mathrm{LS})\int_{[c,d]} f\,dg \in \mathbb{R} \implies$$
  $$\int_c^d f\,dg \in \mathbb{R} \quad\text{and}\quad (\mathrm{LS})\int_{[c,d]} f\,dg = f(c)\,\Delta^- g(c) + \int_c^d f\,dg + f(d)\,\Delta^+ g(d).$$
- $$\int_a^b f\,dg \in \mathbb{R},\ a \le c \le d \le b \implies$$
  $$\int_a^b f\,\chi_{[c,d]}\,dg = f(c)\,\Delta^- g(c) + \int_c^d f\,dg + f(d)\,\Delta^+ g(d).$$

- If $f \in BV[a,b]$, $g \in G[a,b]$, D is the set of discontinuity points of the function f in $[a,b]$ and f^c is continuous part of the function f, $f^c(a) = f(a)$, then

$$\int_a^b f\, dg = \int_a^b f^c\, dg + \sum_D \left[\Delta^- f(d)\,(g(b) - g(d-)) + \Delta^+ f(d)\,(g(b) - g(d+))\right].$$

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- If $f \in G[a,b]$, $g \in BV[a,b]$, D is the set of discontinuity points of the function g in $[a,b]$ and g^c is continuous part of the function g, $g^c(a) = g(a)$, then

$$\int_a^b f\,dg = \int_a^b f\,dg^c + \sum_D f(d)\,\Delta g(d),$$

where $\Delta\,g(a) = \Delta^+\,g(a)$ and $\Delta\,g(b) = \Delta^-\,g(b)$.

5. PROPERTIES OF KS INTEGRAL

ASSUME:

- $f,\ f_k \in G[a,b], \quad g \in BV[a,b]$ for $k \in \mathbb{N}$,
- $f_k \rightrightarrows f$.

THEN: $\displaystyle\int_a^t f_k \, dg \rightrightarrows \int_a^t f \, dg$ on $[a,b]$.

ASSUME:

- $f \in BV[a,b], \quad g, g_k \in G[a,b]$ for $k \in \mathbb{N}$,
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THEN: $\displaystyle\int_a^t f \, dg_k \rightrightarrows \int_a^t f \, dg$ on $[a,b]$.

ASSUME:

- $f,\ f_k \in G[a,b], \quad g, g_k \in BV[a,b]$ for $k \in \mathbb{N}$,
- $f_k \rightrightarrows f, \quad g_k \rightrightarrows g$,
- $\alpha^* := \sup\{\operatorname{var}_a^b g_k : k \in \mathbb{N}\} < \infty$.

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Theorem

ASSUME:

- $f,\, f_k \in G[a,b], \quad g, g_k \in BV[a,b]$ for $k \in \mathbb{N}$,
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THEN: $\displaystyle\int_a^t f_k\, d\,g_k \rightrightarrows \int_a^t f\, d\,g$ on $[a,b]$.

PROOF: Let $\varepsilon > 0$, Choose $k_0 \in \mathbb{N}$ and $\widetilde{\varphi} \in S[a,b]$ in such a way that

$$\|f - \widetilde{\varphi}\|_\infty < \varepsilon/2 \quad \text{and} \quad \|f_k - f\|_\infty < \varepsilon/2,\ \ \|g_k - g\|_\infty < \frac{\varepsilon}{2\,\|\widetilde{\varphi}\|_{BV}} \quad \text{for } k \geq k_0\,.$$

Then $k \geq k_0 \implies \|f_k - \widetilde{\varphi}\|_\infty < \varepsilon$ and

$$\begin{aligned}
&\Big| \int_a^t f_k\, d\,g_k - \int_a^t f\, d\,g \Big| \\
&\quad \leq \Big| \int_a^t (f_k - \widetilde{\varphi})\; d\,g_k \Big| + \Big| \int_a^t \widetilde{\varphi}\; d[g_k - g] \Big| + \Big| \int_a^t (\widetilde{\varphi} - f)\; d\,g \Big| \\
&\quad \leq \|f_k - \widetilde{\varphi}\|_\infty\,(\operatorname{var}_a^b g_k) + 2\,\|\widetilde{\varphi}\|_{BV}\,\|g_k - g\|_\infty + \|\widetilde{\varphi} - f\|_\infty\,(\operatorname{var}_a^b g) \\
&\quad \leq \big(\alpha^* + 1 + \tfrac{1}{2}\,\operatorname{var}_a^b g\big)\,\varepsilon = K\,\varepsilon \quad \text{for every } t \in [a,b]. \qquad \square
\end{aligned}$$

Bounded convergence (Osgood)

ASSUME: $f \in G[a,b]$, $\{f_n\} \subset G[a,b]$ and

- $\|f_n\|_\infty \le M < \infty$ for $n \in \mathbb{N}$,
- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for $x \in [a,b]$.

THEN:

$$\lim_{k\to\infty} \int_a^b f_n \; dg = \int_a^b f \; dg \quad \text{for every } g \in BV[a,b].$$

Standard proof is based on

LEMMA (Arzelà) *Let* $\big\{\{J_{k,j}\} : k \in \mathbb{N},\, j \in U_k\big\}$ *be subintervals of* $[a,b]$ *such that:*

- *for each* $k \in \mathbb{N}$, *the set of indices* U_k *is finite,*
- *the intervals from* $\{J_{k,j} : j \in U_k\}$ *are mutually disjoint,*
- $$\sum_{j \in U_k} |J_{k,j}| > c > 0.$$

Then there exist $\{k_\ell\} \subset \mathbb{N}$ *and* $\{j_\ell\} \subset \mathbb{N}$ *such that*

$$j_\ell \in U_{k_\ell} \quad \text{for } \ell \in \mathbb{N} \quad \text{and} \quad \bigcap_{\ell \in \mathbb{N}} J_{k_\ell, j_\ell} \ne \emptyset.$$

DEFINITIONS

- For intervals $J \subset [a,b]$, the sets $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$ such that
$$\alpha_0 < \alpha_1 < \cdots < \alpha_m \quad \text{and} \quad \alpha_j \in J \quad \text{for } j = 0, 1, \dots, m$$
are **divisions** of J. $\mathcal{D}(J)$ is the set of all divisions of J.
- For $f\colon J \to \mathbb{R}$ we put
$$\mathbf{var}_{\mathbf{J}}\,\mathbf{f} = \sup\{V(f,D) : D \in \mathcal{D}(J)\}, \quad \text{while} \quad \operatorname{var}_{\emptyset} f = \operatorname{var}_{[c]} f = 0 \quad \text{for } c \in [a,b].$$
- A bounded subset E of $\mathbb{R}$ is an **elementary set** if it is a finite union of intervals. For $A \subset \mathbb{R}$, $\mathcal{E}(\mathbf{A})$ is the set of all elementary subsets of A.
- A collection of intervals $\{J_k : k = 1, 2, \dots, m\}$ is a **minimal decomposition** of E if $E = \cup_{k=1}^m J_k$, while $J_k \cup J_\ell$ is not an interval whenever $k \neq \ell$.
- For $f : [a,b] \to X$ and $E \in \mathcal{E}([a,b])$ with a minimal decomposition $\{J_k : k = 1, \dots, m\}$, we define $\mathbf{var}(\mathbf{f}, \mathbf{E}) = \sum_{k=1}^m \operatorname{var}_{J_k} f$.

Proposition

Let $c, d \in [a,b]$, $c < d$. Then

- $\operatorname{var}_{[c,d]} f = \operatorname{var}_c^d f, \qquad \operatorname{var}_{[c,d)} f = \lim_{\delta \to 0+} \operatorname{var}_c^{d-\delta} f = \sup_{t \in [c,d)} \operatorname{var}_c^t f,$
- $\operatorname{var}_{(c,d)} f = \lim_{\delta \to 0+} \operatorname{var}_{c+\delta}^{d-\delta} f, \quad \operatorname{var}_{(c,d]} f = \lim_{\delta \to 0+} \operatorname{var}_{c+\delta}^{d} f = \sup_{t \in (c,d]} \operatorname{var}_t^d f.$
- If $f \in BV((c,d))$ and $f(c+), f(d-)$ exist, then $f \in BV[c,d]$ and
$$\operatorname{var}_c^d f = \operatorname{var}_{(c,d)} f + \|\Delta^+ f(c)\|_X + \|\Delta^- f(d)\|_X.$$

Lewin (1986)

Let $\{A_n\}$ be bounded subsets of $[a, b]$ such that

$$A_{n+1} \subset A_n \quad \text{and} \quad \bigcap A_n = \emptyset.$$

Put

$$\alpha_n = \sup\{\, m(E) : E \in \mathcal{E}(A_n)\} \ \text{ for } \ n \in \mathbb{N}.$$

Then $\lim\limits_{n\to\infty} \alpha_n = 0.$

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LEMMA

Let $f \in BV[a,b] \cap C[a,b]$ and let $\{A_n\} \subset [a,b]$ be bounded and such that

$$A_{n+1} \subset A_n \quad \text{and} \quad \bigcap A_n = \emptyset.$$

Put

$$\alpha_n = \sup\{\, \mathrm{var}(f,E) \colon E \in \mathcal{E}(A_n)\} \;\text{ for }\; n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} \alpha_n = 0.$

DEFINITION

Let $f, g\colon [a,b] \to \mathbb{R}$, and $E \in \mathcal{E}([a,b])$. Then

$$\int_E f\, dg = \int_a^b (f\,\chi_E)\, dg$$

provided the integral on the right-hand side exists.

Propositions

- Let $E_1,\, E_2 \in \mathcal{E}([a,b])$, $E_1 \cap E_2 = \emptyset$, $f, g : [a,b] \to \mathbb{R}$ and let the integrals $\int_{E_j} f\, dg$, $j = 1,2$, exist. Then
$$\int_{E_1 \cup E_2} f\, dg = \int_{E_1} f\, dg + \int_{E_2} f\, dg.$$
- Let $J = (c,d)$ and let $\int_J f\, dg$ exists. Then
$$\Big| \int_J f\, dg \Big| \leq \Big(\operatorname{var}_{(c,d)} g \Big) \Big(\sup_{t \in (c,d)} |g(t)| \Big).$$
- Let $J = [c,d)$, and let $\int_J f\, dg$ and $g(c-)$ exist. Then
$$\Big| \int_J f\, dg \Big| \leq \Big(\operatorname{var}_{[c,d)} g \Big) \Big(\sup_{t \in [c,d)} |g(t)| \Big) + |\Delta^- g(c)|\, |g(c)|.$$

LEMMA

Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

$$\alpha_n = \sup\{\, \mathrm{var}(f,E) \colon E \text{ elementary subset of } A_n \,\} \quad \text{for } n \in \mathbb{N}.$$

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Proof.

LEMMA

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Proof. $\{\alpha_n\}$ is decreasing.

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Proof. $\{\alpha_n\}$ is decreasing. Assume that $\alpha_n \not\to 0$.

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Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Proof. $\{\alpha_n\}$ is decreasing. Assume that $\alpha_n \not\to 0$. Then, there is $\varepsilon > 0$ such that

$$\alpha_n > \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

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Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

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Hence, for each $n \in \mathbb{N}$, there is a closed elementary subset E_n of A_n such that

$$\alpha_n - \frac{\varepsilon}{2^n} < \mathrm{var}(f, E_n).$$

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Define $H_n = \bigcap\limits_{j=1}^{n} E_j$ for $n \in \mathbb{N}$. Then $H_n \subset A_n$ is closed.

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$$\mathrm{var}(f,F) + \mathrm{var}(f,E_n) = \mathrm{var}(f, F \cup E_n) \le \alpha_n \quad \text{for any elementary subset } F \text{ of } A_n \setminus E_n.$$

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Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

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Thus, $\operatorname{var}(f,F) < \varepsilon/2^n$ and since any elementary subset E of $A_n \setminus H_n$ can be written as

$$E = (E \setminus E_1) \cup (E \setminus E_2) \cup \ldots \cup (E \setminus E_n),$$

where $E \setminus E_j$ are elementary subsets of $A_j \setminus E_j$ for $j = 1, \ldots, n$,

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$$\operatorname{var}(f,E) < \varepsilon \quad \text{for every elementary subset } E \text{ of } A_n \setminus H_n.$$

As $\alpha_n > \varepsilon$, this means that there is an elementary subset E of H_n with $\operatorname{var}(f,E) > \varepsilon$.

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Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

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Thus, $\operatorname{var}(f,F) < \varepsilon/2^n$ and since any elementary subset E of $A_n \setminus H_n$ can be written as

$$E = (E \setminus E_1) \cup (E \setminus E_2) \cup \ldots \cup (E \setminus E_n),$$

where $E \setminus E_j$ are elementary subsets of $A_j \setminus E_j$ for $j = 1, \ldots, n$, we get

$$\operatorname{var}(f,E) < \varepsilon \quad \text{for every elementary subset } E \text{ of } A_n \setminus H_n.$$

As $\alpha_n > \varepsilon$, this means that there is an elementary subset E of H_n with $\operatorname{var}(f,E) > \varepsilon$.

Therefore, $H_n \neq \emptyset$ and $\{H_n\}$ are non-empty, closed and bounded sets such that $H_{n+1} \subseteq H_n$.

LEMMA

Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

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$$\alpha_n - \frac{\varepsilon}{2^n} < \operatorname{var}(f,E_n).$$

Define $H_n = \bigcap_{j=1}^{n} E_j$ for $n \in \mathbb{N}$. Then $H_n \subset A_n$ is closed. We will show that $H_n \neq \emptyset$. Obviously,

$$\operatorname{var}(f,F) + \operatorname{var}(f,E_n) = \operatorname{var}(f,F \cup E_n) \le \alpha_n \quad \text{for any elementary subset } F \text{ of } A_n \setminus E_n.$$

Thus, $\operatorname{var}(f,F) < \varepsilon/2^n$ and since any elementary subset E of $A_n \setminus H_n$ can be written as

$$E = (E \setminus E_1) \cup (E \setminus E_2) \cup \ldots \cup (E \setminus E_n),$$

where $E \setminus E_j$ are elementary subsets of $A_j \setminus E_j$ for $j = 1, \ldots, n$, we get

$$\operatorname{var}(f,E) < \varepsilon \text{ for every elementary subset } E \text{ of } A_n \setminus H_n.$$

As $\alpha_n > \varepsilon$, this means that there is an elementary subset E of H_n with $\operatorname{var}(f,E) > \varepsilon$.
Therefore, $H_n \neq \emptyset$ and $\{H_n\}$ are non-empty, closed and bounded sets such that $H_{n+1} \subseteq H_n$.
By Cantor's intersection theorem we get $\bigcap_n H_n \neq \emptyset$.

LEMMA

Let $f \in BV[a,b]$ be continuous on $[a,b]$ and let $\{A_n\}$ be bounded subsets of $[a,b]$ such that $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Put

$$\alpha_n = \sup\{\operatorname{var}(f,E) \colon E \text{ elementary subset of } A_n\} \quad \text{for } n \in \mathbb{N}.$$

Then $\lim\limits_{n\to\infty} \alpha_n = 0.$

Proof. $\{\alpha_n\}$ is decreasing. Assume that $\alpha_n \not\to 0$. Then, there is $\varepsilon > 0$ such that

$$\alpha_n > \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

Hence, for each $n \in \mathbb{N}$, there is a closed elementary subset E_n of A_n such that

$$\alpha_n - \frac{\varepsilon}{2^n} < \operatorname{var}(f, E_n).$$

Define $H_n = \bigcap\limits_{j=1}^{n} E_j$ for $n \in \mathbb{N}$. Then $H_n \subset A_n$ is closed. We will show that $H_n \neq \emptyset$. Obviously,

$$\operatorname{var}(f,F) + \operatorname{var}(f,E_n) = \operatorname{var}(f, F \cup E_n) \le \alpha_n \quad \text{for any elementary subset } F \text{ of } A_n \setminus E_n.$$

Thus, $\operatorname{var}(f,F) < \varepsilon/2^n$ and since any elementary subset E of $A_n \setminus H_n$ can be written as

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$$\operatorname{var}(f,E) < \varepsilon \quad \text{for every elementary subset } E \text{ of } A_n \setminus H_n.$$

As $\alpha_n > \varepsilon$, this means that there is an elementary subset E of H_n with $\operatorname{var}(f,E) > \varepsilon$.

Therefore, $H_n \neq \emptyset$ and $\{H_n\}$ are non-empty, closed and bounded sets such that $H_{n+1} \subseteq H_n$.

By Cantor's intersection theorem we get $\bigcap_n H_n \neq \emptyset$.

This contradicts our assumption $\bigcap_n A_n = \emptyset$ and hence, $\lim\limits_{n\to\infty} \alpha_n = 0$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a, b]$.

a) $(\operatorname{var}_a^b g = 0) \Rightarrow \int_a^b f_n \, dg = \int_a^b f \, dg = 0$ for all $n \in \mathbb{N}$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

a) $(\operatorname{var}_a^b g = 0) \Rightarrow \int_a^b f_n \, dg = \int_a^b f \, dg = 0$ for all $n \in \mathbb{N}$.

b) Let $\operatorname{var}_a^b g \neq 0$, $\varepsilon > 0$ and $A_n = \left\{ t \in [a,b] : \exists m \ge n \text{ such that } \|f_n(t)\|_\infty \ge \dfrac{\varepsilon}{6 \operatorname{var}_a^b g} \right\}$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \leq K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\operatorname{var}(g, E) : E \in \mathcal{E}(A_n)\} \searrow 0$ due to LEMMA.

Sketch of the proof of Bounded Convergence Theorem

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a, b]$.

a) $(\,\mathrm{var}_a^b\, g = 0\,) \Rightarrow \int_a^b f_n\, dg = \int_a^b f\, dg = 0$ for all $n \in \mathbb{N}$.

b) Let $\mathrm{var}_a^b\, g \neq 0$, $\varepsilon > 0$ and $A_n = \left\{ t \in [a, b] : \exists m \ge n \text{ such that } \|f_n(t)\|_\infty \ge \dfrac{\varepsilon}{6\, \mathrm{var}_a^b\, g} \right\}$.

Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\mathrm{var}(g, E) : E \in \mathcal{E}(A_n)\} \searrow 0$ due to LEMMA.
Hence, $\alpha_n < \dfrac{\varepsilon}{6\,K}$ for $n \ge N$, i.e.

(1) $\quad \mathrm{var}(g, E) < \dfrac{\varepsilon}{6\,K}$ for $E \in \mathcal{E}(A_n)$ and $n \ge N$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

a) $(\operatorname{var}_a^b g = 0) \Rightarrow \int_a^b f_n \, d g = \int_a^b d g \, g = 0$ for all $n \in \mathbb{N}$.

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup \{\operatorname{var}(g, E) : E \in \mathcal{E}(A_n)\} \searrow 0$ and

(1) $\operatorname{var}(g, E) < \dfrac{\varepsilon}{6\,K}$ for $E \in \mathcal{E}(A_n)$ and $n \ge N$.

Sketch of the proof of Bounded Convergence Theorem

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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(1) $\operatorname{var}(g,E) < \dfrac{\varepsilon}{6K}$ for $E \in \mathcal{E}(A_n)$ and $n \ge N$.

Let $n \ge N$ and let $h_n \in S$ be such that $\|h_n - f_n\|_\infty < \min\left\{ K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b g} \right\}$.

Denote $U_n = \left\{ t \in [a,b] : |h_n(t)| \ge \dfrac{\varepsilon}{3 \operatorname{var}_a^b g} \right\}$ and $V_n = [a,b] \setminus U_n$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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We have

$$t \in U_n \Rightarrow |f_n(t)| > |h_n(t)| - \frac{\varepsilon}{6\,\mathrm{var}_a^b\, g} \ge \frac{\varepsilon}{3\,\mathrm{var}_a^b\, g} - \frac{\varepsilon}{6\,\mathrm{var}_a^b\, g} = \frac{\varepsilon}{6\,\mathrm{var}_a^b\, g}$$

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

a) $(\,\mathrm{var}_a^b\, g = 0\,) \Rightarrow \int_a^b f_n\, dg = \int_a^b dg\, g = 0$ for all $n \in \mathbb{N}$.

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\mathrm{var}(g,E) : E \in \mathcal{E}(A_n)\} \searrow 0$ and

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Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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i.e. $U_n \subset A_n$.

Let $g \in BV \cap C,\ \|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\operatorname{var}(g,E) : E \in \mathcal{E}(A_n)\} \searrow 0$ and

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We have $U_n \subset A_n$. Hence, by (1),

$$\left| \int_a^b dg\, h_n \right| \le \left| \int_{U_n} dg\, h_n \right| + \left| \int_{V_n} dg\, h_n \right| \le \operatorname{var}(g,U_n)\, \|h_n\|_{U_n} + \operatorname{var}(g,V_n)\, \|h_n\|_{V_n}$$

$$\le \frac{\varepsilon}{6\,K}(K+K) + \operatorname{var}_a^b g \, \frac{\varepsilon}{3 \operatorname{var}_a^b g} = \frac{2}{3}\,\varepsilon$$

Sketch of the proof of Bounded Convergence Theorem

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

a) $(\operatorname{var}_a^b g = 0) \Rightarrow \int_a^b f_n \, d g = \int_a^b d g \, g = 0$ for all $n \in \mathbb{N}$.

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\operatorname{var}(g,E) : E \in \mathcal{E}(A_n)\} \searrow 0$ and

(1) $\operatorname{var}(g,E) < \dfrac{\varepsilon}{6\,K}$ for $E \in \mathcal{E}(A_n)$ and $n \ge N$.

Let $n \ge N$ and let $h_n \in S$ be such that $|h_n - f_n| < \min\left\{K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b g}\right\}$.

We have $\left| \int_a^b d g \, h_n \right| < \dfrac{2}{3}\,\varepsilon$.

Let $g \in BV \cap C$, $\|f_n\|_\infty \leq K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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(1) $\operatorname{var}(g,E) < \dfrac{\varepsilon}{6K}$ for $E \in \mathcal{E}(A_n)$ and $n \geq N$.

Let $n \geq N$ and let $h_n \in S$ be such that $|h_n - f_n| < \min\left\{K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b g}\right\}$.

We have $\left| \int_a^b dg \, h_n \right| < \dfrac{2}{3} \varepsilon$. Therefore,

$$\left| \int_a^b f_n \, dg \right| \leq \left| \int_a^b dg \, h_n \right| + \left| \int_a^b dg \, (h_n - f_n) \right| \leq \frac{2}{3} \varepsilon + (\operatorname{var}_a^b g) \frac{\varepsilon}{6 \operatorname{var}_a^b g} < \varepsilon .$$

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

a) $(\operatorname{var}_a^b g = 0) \Rightarrow \int_a^b f_n \, dg = \int_a^b dg\, g = 0$ for all $n \in \mathbb{N}$.

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Let $n \ge N$ and let $h_n \in S$ be such that $|h_n - f_n| < \min\left\{K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b g}\right\}$.

We have $\left| \int_a^b dg\, h_n \right| < \dfrac{2}{3}\varepsilon$. Therefore,

$$\left| \int_a^b f_n \, dg \right| \le \left| \int_a^b dg\, h_n \right| + \left| \int_a^b dg\, (h_n - f_n) \right| \le \frac{2}{3}\varepsilon + (\operatorname{var}_a^b g)\, \frac{\varepsilon}{6 \operatorname{var}_a^b g} < \varepsilon\,.$$

If $g \in BV \setminus C$,

Let $g \in BV \cap C$, $\|f_n\|_\infty \le K < \infty$ for $n \in \mathbb{N}$ and $f_n(t) \to 0$ on $[a,b]$.

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Let $n \ge N$ and let $h_n \in S$ be such that $|h_n - f_n| < \min\left\{ K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b g} \right\}$.

We have $\left| \int_a^b d g\, h_n \right| < \dfrac{2}{3}\varepsilon$. Therefore,

$$\left| \int_a^b f_n \, dg \right| \le \left| \int_a^b d g\, h_n \right| + \left| \int_a^b d g\,(h_n - f_n) \right| \le \frac{2}{3}\varepsilon + (\operatorname{var}_a^b g)\, \frac{\varepsilon}{6 \operatorname{var}_a^b g} < \varepsilon\,.$$

If $g \in BV \setminus C$, we split $g = g_{cont} + g_{jump}$ .... □

Integration by parts

Let $f \in G[a,b]$, $g \in BV[a,b]$. Then both integrals

$$\int_a^b f\,dg \quad \text{and} \quad \int_a^b g\,df$$

exist and it holds

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)\,g(b) - f(a)\,g(a) - \sum_{a \le t < b} \Delta^+ f(t)\,\Delta^+ g(t) + \sum_{a < t \le b} \Delta^- f(t)\,\Delta^- g(t)\,.$$

Substitution

Let $h \in BV[a,b]$, $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are such that $\int_a^b f\,dg$ exists.

Then, if one from the integrals

$$\int_a^b h(t)\,d\Big[\int_a^t f\,dg\Big], \quad \int_a^b h\,f\,dg\,,$$

exists, the same is true also for the remaining one and

$$\int_a^b h(t)\,d\Big[\int_a^t f\,dg\Big] = \int_a^b h\,f\,dg\,.$$

The Saks-Henstock lemma is an indispensable tool in the study of deeper properties of the Kurzweil-Stieltjes integral.

Saks-Henstock Lemma

ASSUME: $\int_a^b f\,dg$ exists, $\varepsilon > 0$ is given and δ_ε is a gauge on $[a,b]$ such that

$$\left| S(P) - \int_a^b f\,dg \right| < \varepsilon \text{ for all } \delta_\varepsilon\text{-fine partitions } P \text{ of } [a,b],$$

THEN:

$$\left| \sum_{j=1}^n \left(f(\theta_j)\,(g(t_j) - g(s_j)) - \int_{s_j}^{t_j} f\,dg \right) \right| \le \varepsilon$$

holds for every system $\{([s_j,t_j],\theta_j) : j \in \{1,\dots,n\}\}$ such that

$$a \le s_1 \le \theta_1 \le t_1 \le s_2 \le \cdots \le s_n \le \theta_n \le t_n \le b,$$

and

$$[s_j,t_j] \subset (\theta_j - \delta(\theta_j), \theta_j + \delta(\theta_j)) \quad \text{for } j \in \{1,\dots,n\}.$$

Corollaries

- If $\int_a^b f\ dg$ exists, $\varepsilon > 0$ is given and δ_ε is a gauge on $[a, b]$ such that
$$\left| S(P) - \int_a^b f\ dg \right| < \varepsilon \text{ for all } \delta_\varepsilon\text{-fine partitions } P \text{ of } [a, b],$$
then
$$\sum_{j=1}^{\nu(P)} \left| f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})] - \int_{\alpha_{j-1}}^{\alpha_j} f\ dg \right| \leq \varepsilon$$
holds for every δ_ε-fine partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a, b]$.

Corollaries

- If $\int_a^b f\,dg$ exists, $\varepsilon > 0$ is given and δ_ε is a gauge on $[a,b]$ such that
$$\left|S(P) - \int_a^b f\,dg\right| < \varepsilon \text{ for all } \delta_\varepsilon\text{-fine partitions } P \text{ of } [a,b],$$
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- If $f \in G[a,b]$, $g \in G[a,b]$ and at least one of them has a bounded variation, then
$$h(t) = \int_a^t f\,dg \quad \text{is regulated on } [a,b]\,.$$
In particular, if $g \in BV[a,b]$, then also $h \in BV[a,b]$.

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In particular, if $g \in BV[a,b]$, then also $h \in BV[a,b]$.

- $\Delta^+ h(t) = f(t)\,\Delta^+ g(t)$ for $t \in [a,b)$, $\quad \Delta^- h(s) = f(s)\,\Delta^- g(s)$ for $s \in (a,b]$.

Theorem (Hake)

- $\int_a^t f \; dg$ exists for every $t \in [a, b)$ and $\lim\limits_{t \to b-} \left(\int_a^t f \; dg + f(b) \, [g(b) - g(t)] \right) = I \in \mathbb{R}$
$$\implies \int_a^b f \; dg = I.$$
- $\int_t^b f \; dg$ exists for every $t \in (a, b\,]$ and $\lim\limits_{t \to a+} \left(\int_t^b f \; dg + f(a) \, [g(t) - g(a)] \right) = I \in \mathbb{R}$
$$\implies \int_a^b f \; dg = I.$$

6. CONTINUOUS LINEAR FUNCTIONALS

Riesz Theorem

Φ is continuous linear functional on $C[a,b]$ $\left(\Phi \in (C[a,b])^*\right)$ $\Leftrightarrow$
there is $p \in BV[a,b]$ such that $p\,(a) = 0$, p is right continuous on (a,b) $(p \in NBV[a,b])$ and

$$\Phi(x) = \Phi_p(x) := \int_a^b x\ dp \quad \text{for every } \ x \in C[a,b].$$

Mapping $p \in NBV[a,b] \to \Phi_p \in (C[a,b])^*$ is isometric isomorphism.

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$$G_L[a,b] = \{x \in G[a,b] : x(t-) = x(t) \text{ for } t \in (a,b)\}$$

Continuous linear functionals on the space $G_L[a,b]$

Φ is continuous linear functional on $G_L[a,b]$ $(\Phi \in (G_L[a,b])^*)$ $\Leftrightarrow$
exist $p \in BV[a,b]$ and $q \in \mathbb{R}$ such that

$$\Phi(x) = \Phi_{(p,q)}(x) := q\, x(a) + \int_a^b p \, dx \quad \text{for } x \in G_L[a,b].$$

Mapping $(p,q) \in BV[a,b] \times \mathbb{R} \to \Phi_{(p,q)} \in (G_L[a,b])^*$ is isomorphism.

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Mapping $p \in NBV[a,b] \to \Phi_p \in (C[a,b])^*$ is isometric isomorphism.

$$\widetilde{G}_L[a,b] = \{x \in G[a,b] : x(t-) = x(t) \text{ for } t \in (a,b]\}$$

Continuous linear functionals on the space $\widetilde{G}_L[a,b]$

Φ is continuous linear functional on $\widetilde{G}_L[a,b]$ $(\Phi \in (\widetilde{G}_L[a,b])^*)$ $\Leftrightarrow$
there is $p \in BV[a,b]$ such that

$$\Phi(x) = \Phi_p(x) := p(b)\,x(b) - \int_a^b p\, dx \quad \text{for } x \in \widetilde{G}_L[a,b].$$

Mapping $p \in BV[a,b] \to \Phi_p \in (G_L[a,b])^*$ is isomorphism.

7. Generalized linear differential equations

(I) $\quad x' = P(t)\,x + q(t), \quad \Delta^+ x(\tau_k) = B_k\, x(\tau_k) + d_k\,, \quad k = 1, 2, \dots, r\,,$

where $\quad a = t_0 < t_1 < \dots < t_r = b,$

$$P \in L^1([a,b], \mathbb{R}^{n\times n}),\ q \in L^1([a,b], \mathbb{R}^n),\ B_k \in \mathbb{R}^{n\times n},\ d_k \in \mathbb{R}^n.$$

$$\tau \in (a,b),\ B \in \mathbb{R}^{n\times n} \implies \int_a^b d[\chi_{(\tau,b]}(s)\,B]\,x(s) = B\,x(\tau)$$

Define

$$\left.\begin{aligned} A(t) &= \int_a^t P(s)\,ds + \sum_{k=1}^r \chi_{(\tau_k,b]}(t)\,B_k\,, \\ f(t) &= \int_a^t q(s)\,ds + \sum_{k=1}^r \chi_{(\tau_k,b]}(t)\,d_k \end{aligned}\right\} \quad \text{for} \ \ t \in [a,b]\,.$$

Then

$$\text{(I)} \quad \Leftrightarrow \quad x(t) = x(a) + \int_a^t dA\,x + f(t) - f(a), \quad t \in [a,b],$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d A\, x + f(t) - f(t_0)\,, \quad t \in [a,b] \qquad \big[A \in BV([a,b], \mathbb{R}^{n\times n})\big]\,.$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(t_0)\,, \quad t \in [a,b] \qquad \big[A \in BV([a,b],\mathbb{R}^{n\times n})\big].$$

Operator $\quad (L\,x)(t) = \displaystyle\int_{t_0}^{t} d\,A\,x \quad$ is linear and compact on $BV([a,b],\mathbb{R}^n) \implies$

by FREDHOLM ALTERNATIVE we have

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(t_0)\,, \quad t \in [a,b] \qquad \big[A \in BV([a,b],\mathbb{R}^{n\times n})\big].$$

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Lemma

(L) has 1! and solution for each $f \in BV([a,b],\mathbb{R}^n)$ iff the homogeneous equation

$$\text{(H)} \qquad x(t) = \int_{t_0}^{t} d\,A\,x$$

has only trivial solution.

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Lemma

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has only trivial solution.

Lemma

Let

$$\det\big[I - \Delta^- A(t)\big] \neq 0 \;\text{ and }\; \det\big[I + \Delta^+ A(s)\big] \neq 0 \quad \text{for each } t \in (t_0, b\,] \text{ and each } s \in [a, t_0)\,.$$

Then (H) has only trivial solution.

Sketch of proof

- $\Delta^+ x(t_0) = \Delta^+ A(t_0)\, x(t_0) = 0 \implies x(t_0+) = 0.$

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|x(t)| &\le \int_{t_0}^{t} d[\alpha]\, x = \Delta^+\alpha(t_0)\, |x(t_0)| + \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \\
&= \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \le \big[\alpha(t_0+\delta) - \alpha(t_0+)\big] \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big) \\
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- $\Delta^+ x(t_0) = \Delta^+ A(t_0)\, x(t_0) = 0 \implies x(t_0+) = 0.$
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Sketch of proof

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- $x = 0$ on $[t_0, t_0 + \delta]$.
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- $t^* < b \implies x(t^*+) = \Delta^*(t)\, x(t^*) = 0$

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- $\alpha(t) = \operatorname{var}_{t_0}^{t} A.$
- Choose $\delta \in (0, b - t_0)$ so that $0 \le \alpha(t_0 + \delta) - \alpha(t_0+) < 1/2.$
- For $t \in [t_0, t_0{+}\delta]$ we have
$$|x(t)| \le \int_{t_0}^{t} d[\alpha]\, x = \Delta^+\alpha(t_0)\,|x(t_0)| + \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\,|x|$$
$$= \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\,|x| \le \left[\alpha(t_0{+}\delta) - \alpha(t_0+)\right] \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big)$$
$$\le \frac{1}{2} \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big).$$
- $\Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big) \le \frac{1}{2} \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big).$
- $x = 0$ on $[t_0, t_0 + \delta].$
- $t^* = \sup\{\tau \in (t_0, b\,] : x = 0 \text{ on } [t_0, \tau]\}$
- $x = 0$ on $[t_0, t^*) \implies x(t^*-) = 0 \implies 0 = \left[I - \Delta^- A(t^*)\right] x(t^*) \implies x(t^*) = 0$
  $\implies x(t) = 0$ on $[0, t^*].$
- $t^* < b \implies x(t^*+) = \Delta^*(t)\, x(t^*) = 0$
  $\implies x(t) = 0$ on $[0, t^* + \delta]$ for some $\delta \in (0, b - t^*)$

Sketch of proof

- $\Delta^+ x(t_0) = \Delta^+ A(t_0)\, x(t_0) = 0 \implies x(t_0+) = 0.$
- $\alpha(t) = \operatorname{var}_{t_0}^{t} A.$
- Choose $\delta \in (0, b - t_0)$ so that $0 \le \alpha(t_0 + \delta) - \alpha(t_0+) < 1/2.$
- For $t \in [t_0, t_0 + \delta]$ we have

$$\begin{aligned} |x(t)| &\le \int_{t_0}^{t} d[\alpha]\, x = \Delta^+ \alpha(t_0)\, |x(t_0)| + \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \\ &= \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \le \big[\alpha(t_0+\delta) - \alpha(t_0+)\big] \left(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \right) \\ &\le \frac{1}{2} \left(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \right). \end{aligned}$$

- $\left(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \right) \le \frac{1}{2} \left(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \right).$
- $x = 0$ on $[t_0, t_0 + \delta].$
- $t^* = \sup\{\tau \in (t_0, b] : x = 0 \text{ on } [t_0, \tau]\}$
- $x = 0$ on $[t_0, t^*) \implies x(t^*-) = 0 \implies 0 = \big[I - \Delta^- A(t^*)\big]\, x(t^*) \implies x(t^*) = 0$ $\implies x(t) = 0$ on $[0, t^*].$
- $t^* < b \implies x(t^*+) = \Delta^*(t)\, x(t^*) = 0$ $\implies x(t) = 0$ on $[0, t^* + \delta]$ for some $\delta \in (0, b - t^*) \implies$ CONTRADICTION

- $\Delta^+ x(t_0) = \Delta^+ A(t_0)\, x(t_0) = 0 \implies x(t_0+) = 0.$
- $\alpha(t) = \operatorname{var}_{t_0}^{t} A.$
- Choose $\delta \in (0, b - t_0)$ so that $0 \le \alpha(t_0 + \delta) - \alpha(t_0+) < 1/2.$
- For $t \in [t_0, t_0+\delta]$ we have

$$\begin{aligned}
|x(t)| &\le \int_{t_0}^{t} d[\alpha]\, x = \Delta^+\alpha(t_0)\, |x(t_0)| + \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \\
&= \lim_{\tau \to t_0+} \int_{\tau}^{t} d[\alpha]\, |x| \le \big[\alpha(t_0+\delta) - \alpha(t_0+)\big] \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big) \\
&\le \frac{1}{2} \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)| \Big).
\end{aligned}$$

- $\Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)|\Big) \le \frac{1}{2} \Big(\sup_{t \in [t_0, t_0+\delta]} |x(t)|\Big).$
- $x = 0$ on $[t_0, t_0 + \delta].$
- $t^* = \sup\{\tau \in (t_0, b\,] : x = 0 \text{ on } [t_0, \tau]\}$
- $x = 0$ on $[t_0, t^*) \implies x(t^*-) = 0 \implies 0 = \big[I - \Delta^- A(t^*)\big]\, x(t^*) \implies x(t^*) = 0$
  $\implies x(t) = 0$ on $[0, t^*].$
- $t^* < b \implies x(t^*+) = \Delta^*(t)\, x(t^*) = 0$
  $\implies x(t) = 0$ on $[0, t^* + \delta]$ for some $\delta \in (0, b - t^*) \implies$ CONTRADICTION
  $\implies x \equiv 0$ on $[t_0, b].$ $\square$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

Theorem

ASSUME:

- $A \in BV([a,b], \mathbb{R}^{n\times n})$ and $t_0 \in [a,b]$.
- $$\det\big[I - \Delta^- A(t)\big] \neq 0 \ \text{ for each } \ t \in (t_0, b\,],$$
  $$\det\big[I + \Delta^+ A(s)\big] \neq 0 \ \text{ for each } \ s \in [a, t_0).$$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

Theorem

<u>ASSUME</u>:

- $A \in BV([a,b],\mathbb{R}^{n\times n})$ and $t_0 \in [a,b]$.
- $$\det\big[I - \Delta^{-}A(t)\big] \neq 0 \ \text{ for each } \ t \in (t_0, b\,],$$ $$\det\big[I + \Delta^{+}A(s)\big] \neq 0 \ \text{ for each } \ s \in [a, t_0).$$

<u>THEN</u>: for each $f \in BV([a,b],\mathbb{R}^n)$ and $\widetilde{x} \in X$, (L) has 1! solution $x \in BV([a,b],\mathbb{R}^n)$.

Gronwall lemma

ASSUME: $u, p : [a,b] \to [0,\infty)$ continuous, $K, L \geq 0$ and $u(t) \leq K + L\int_a^t (p\, u)\, ds$ for $t \in [a,b]$.

THEN: $u(t) \leq K\ \exp\big(L \int_a^t p\, ds\big)$ for $t \in [a,b]$.

Generalized Gronwall lemma

ASSUME:

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuous on $(a,b\,]$,
- $u(t) \leq K + L \int_a^t u\, dh$ for $t \in [a,b]$.

THEN: $u(t) \leq K\ \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.

Corollary

ASSUME: $A \in BV([a,b], \mathbb{R}^{n\times n})$, $f \in G([a,b], \mathbb{R}^n)$, $\det\big[I - \Delta^- A(t)\big] \neq 0$ for $t \in (a,b\,]$ and

$$c_A = \sup\{|[I - \Delta^- A(t)]^{-1}| : t \in [a,b)\}.$$

THEN: $0 < c_A < \infty$ and $|x(t)| \leq c_A \Big(|\widetilde{x}| + 2\,\|f\|_\infty\Big) \exp(2\, c_A\ \mathrm{var}_a^t A)$ on $[a,b]$

holds for every solution x of the equation

$$x(t) = \widetilde{x} + \int_a^t d\,A\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

Assumptions

- $u : [a, b] \to [0, \infty)$ is bounded on $[a, b]$, $K, L \geq 0$,
- $h : [a, b] \to [0, \infty)$ is nondecreasing and left-continuous on $(a, b\,]$,
- $u(t) \leq K + L \displaystyle\int_a^t u\, dh$ for $t \in [a, b]$.

- $\kappa \geq 0 \to w_\kappa(t) = \kappa \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a, b]$.
-
$$\int_a^t w_\kappa\, dh = \kappa \int_a^t \exp\big(L\,[h(s) - h(a)]\big)\; dh(s)]$$
$$= \kappa \int_a^t \Big(\sum_{k=0}^{\infty} \frac{L^k}{k!}\,[h(s) - h(a)]^k\Big)\; dh(s)] = \kappa \sum_{k=0}^{\infty} \Big(\frac{L^k}{k!} \int_a^t \big[h(s) - h(a)\big]^k\Big)\; dh(s)]$$
$$\leq \kappa \sum_{k=0}^{\infty} \Big(\frac{L^k\,[h(t) - h(a)]^{k+1}}{(k+1)!}\Big) = \frac{\kappa}{L}\,\Big(\exp(L\,[h(t) - h(a)]) - 1\Big)$$
$$= \frac{w_\kappa(t) - \kappa}{L} \quad \text{for } \ t \in [a, b].$$

Assumptions

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuous on $(a,b\,]$,
- $u(t) \leq K + L \displaystyle\int_a^t u\,d\,h$ for $t \in [a,b]$.

- $\kappa \geq 0 \to w_\kappa(t) = \kappa \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.
- $\displaystyle\int_a^t w_\kappa\,d\,h \leq \frac{w_\kappa(t) - \kappa}{L}$ for $t \in [a,b] \implies$

  $w_\kappa(t) \geq \kappa + L \displaystyle\int_a^t w_\kappa\,d\,h$ for every $\kappa \geq 0$ and $t \in [a,b]$.

Assumptions

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuous on $(a,b\,]$,
- $u(t) \leq K + L \displaystyle\int_a^t u\,d\,h$ for $t \in [a,b]$.

- $\kappa \geq 0 \to w_\kappa(t) = \kappa\,\exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.
- $w_\kappa(t) \geq \kappa + L \displaystyle\int_a^t w_\kappa\,d\,h$ for every $\kappa \geq 0$ and $t \in [a,b]$.

Assumptions

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuous on $(a,b\,]$,
- $u(t) \leq K + L \displaystyle\int_a^t u\,d\,h$ for $t \in [a,b]$.

- $\kappa \geq 0 \to w_\kappa(t) = \kappa \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.
- $w_\kappa(t) \geq \kappa + L \displaystyle\int_a^t w_\kappa\,d\,h$ for every $\kappa \geq 0$ and $t \in [a,b]$.
- Let $\varepsilon > 0$ be given. Put $\kappa = K + \varepsilon$ and $v_\varepsilon = u - w_\kappa$.
- Subtracting the blue inequalities we find out
$$v_\varepsilon(t) \leq -\varepsilon + L \int_a^t v_\varepsilon\,d\,h \quad \text{for } t \in [a,b]$$
wherefrom, using Hake Theorem twice, one can deduce that $v_\varepsilon < 0$ on $[a,b]$.

Assumptions

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuous on $(a,b\,]$,
- $u(t) \leq K + L \displaystyle\int_a^t u\,dh$ for $t \in [a,b]$.

- $\kappa \geq 0 \to w_\kappa(t) = \kappa \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.
- $w_\kappa(t) \geq \kappa + L \displaystyle\int_a^t w_\kappa\,d\,h$ for every $\kappa \geq 0$ and $t \in [a,b]$.
- Let $\varepsilon > 0$ be given. Put $\kappa = K + \varepsilon$ and $v_\varepsilon = u - w_\kappa$.
- Subtracting the blue inequalities we find out

$$v_\varepsilon(t) \leq -\varepsilon + L \int_a^t v_\varepsilon\,d\,h \quad \text{for } t \in [a,b]$$

wherefrom, using Hake Theorem twice, one can deduce that $v_\varepsilon < 0$ on $[a,b]$.

Therefore

$$u(t) < w_\kappa(t) = K \exp\big(L\,(h(t)-h(a))\big) + \varepsilon \exp\big(L\,(h(t)-h(a))\big) \quad \text{for } t \in [a,b].$$

Since $\varepsilon > 0$ could be arbitrary, this proves Lemma. $\square$

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} dA\,x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

$$(\mathrm{L}) \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d A\, x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

Corollary

ASSUME:

- $A \in BV([a,b],\mathbb{R}^{n\times n})$ and $t_0 \in [a,b]$.
- $\det\,[I - \Delta^{-}A(t)] \neq 0$ for $t \in (t_0, b\,]$,

  $\det\,[I + \Delta^{+}A(s)] \neq 0$ for $s \in [a, t_0)$.

THEN: for each $f \in G([a,b],\mathbb{R}^n)$ and $\widetilde{x} \in \mathbb{R}^n$, (L) has 1! solution $x \in G([a,b],\mathbb{R}^n)$.

Continuous dependence

$$x_k(t) = \widetilde{x}_k + \int_a^t d\,[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d\,[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

Continuous dependence

$$x_k(t) = \widetilde{x}_k + \int_a^t d\,[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d\,[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

$A_k, A \in BV([a,b], \mathbb{R}^{n\times n}), \quad f_k, f \in G([a,b], \mathbb{R}^n), \quad \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n \quad$ for $k \in \mathbb{N}$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d\,[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d\,[A]\,x \; + f(t) - f(a), \qquad t \in [a,b].$$

$$A_k, A \in BV([a,b], \mathbb{R}^{n\times n}), \quad f_k, f \in G([a,b], \mathbb{R}^n), \quad \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n \quad \text{for } k \in \mathbb{N}\,.$$

Theorem

ASSUME:

- $\det\,[I - \Delta^{-} A(t)] \neq 0 \quad \text{for } \; t \in (a,b],$
- $A_k \rightrightarrows A \quad \text{on } \; [a,b], \quad \alpha^* := \sup\{\operatorname{var}_a^b A_k : k \in \mathbb{N}\} < \infty,$
- $\widetilde{x}_k \to \widetilde{x}, \quad f_k \rightrightarrows f \quad \text{on } \; [a,b].$

THEN: $x_k \rightrightarrows x \quad \text{on } \; [a,b].$

Sketch of the proof

WE ASSUME:

- $A_k, A \in BV([a,b], \mathbb{R}^{n\times n}),\ f_k, f \in G([a,b], \mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N}$, are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k \,:\, k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

WE ASSUME:

- $A_k, A \in BV([a,b], \mathbb{R}^{n\times n})$, $f_k, f \in G([a,b], \mathbb{R}^n)$, $\widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- A_k, $k \in \mathbb{N}$, are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\operatorname{var}_a^b A_k \,:\, k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

SHOW that there is k_0 such that $\det[I - \Delta^- A_k(t)] \neq 0$ and $c_{A_k} \le 2\, c_A$ for $k \ge k_0$ and restrict hereafter to $k \ge k_0$.

WE ASSUME:

- $A_k, A \in BV([a,b], \mathbb{R}^{n\times n})$, $f_k, f \in G([a,b], \mathbb{R}^n)$, $\widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- A_k, $k \in \mathbb{N}$, are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

SHOW that there is k_0 such that $\det[I - \Delta^- A_k(t)] \neq 0$ and $c_{A_k} \leq 2\,c_A$ for $k \geq k_0$ and restrict hereafter to $k \geq k_0$.

PUT $\quad w_k = (x_k - f_k) - (x - f)$.

WE ASSUME:

- $A_k, A \in BV([a,b],\mathbb{R}^{n\times n}),\ f_k, f \in G([a,b],\mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N},$ are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

SHOW that there is k_0 such that $\det[I - \Delta^- A_k(t)] \neq 0$ and $c_{A_k} \leq 2\,c_A$ for $k \geq k_0$ and restrict hereafter to $k \geq k_0$.

PUT $w_k = (x_k - f_k) - (x - f)$.

THEN

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\,w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

WE ASSUME:

- $A_k, A \in BV([a,b],\mathbb{R}^{n\times n}),\ f_k, f \in G([a,b],\mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N},$ are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

SHOW that there is k_0 such that $\det[I - \Delta^- A_k(t)] \neq 0$ and $c_{A_k} \leq 2\, c_A$ for $k \geq k_0$ and restrict hereafter to $k \geq k_0$.

PUT $w_k = (x_k - f_k) - (x - f)$.

THEN

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

where

$$\widetilde{w}_k = (\widetilde{x}_k - f_k(a)) - (\widetilde{x} - f(a)) \to 0, \quad h_k(t) = \int_a^t d[A_k - A]\,(x - f) + \Big(\int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f \Big),$$

$$\lim_{k\to\infty} \Big\| \int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f \Big\|_{\mathbb{R}^n} = 0 \quad \text{for } t \in [a,b]$$

$$\Big\| \int_a^t d[A_k - A](x - f) \Big\|_{\mathbb{R}^n} \leq 2\, \|A_k - A\|_\infty\, \|x - f\|_{BV} \quad \text{on } [a,b] \quad (\text{since } (x-f) \in BV([a,b],\mathbb{R}^{n\times n})).$$

WE ASSUME:

- $A_k, A \in BV([a,b],\mathbb{R}^{n\times n}),\ f_k, f \in G([a,b],\mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N},$ are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

SHOW that there is k_0 such that $\det[I - \Delta^- A_k(t)] \neq 0$ and $c_{A_k} \leq 2\, c_A$ for $k \geq k_0$ and restrict hereafter to $k \geq k_0$.

PUT $w_k = (x_k - f_k) - (x - f)$.

THEN

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

where

$$\widetilde{w}_k = (\widetilde{x}_k - f_k(a)) - (\widetilde{x} - f(a)) \to 0, \quad h_k(t) = \int_a^t d[A_k - A]\,(x - f) + \Big(\int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f\Big),$$

$$\lim_{k\to\infty} \Big\| \int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f \Big\|_{\mathbb{R}^n} = 0 \quad \text{for } t \in [a,b]$$

$$\Big\| \int_a^t d[A_k - A](x - f) \Big\|_{\mathbb{R}^n} \leq 2\, \|A_k - A\|_\infty\, \|x - f\|_{BV} \quad \text{on } [a,b] \quad (\text{since } (x - f) \in BV([a,b],\mathbb{R}^{n\times n})).$$

SUMMARIZED: $\lim_{k\to\infty} \|h_k\|_\infty = 0$, $\lim_{k\to\infty} \widetilde{w}_k = 0$.

WE ASSUME:

- $A_k, A \in BV([a,b],\mathbb{R}^{n\times n}),\ f_k, f \in G([a,b],\mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N},$ are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k\ :\ k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

WE HAVE: $w_k = (x_k - f_k) - (x - f)$,

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b]\,,$$

$$\lim_{k\to\infty} \|h_k\|_\infty = 0, \qquad \lim_{k\to\infty} \widetilde{w}_k = 0.$$

Sketch of the proof

WE ASSUME:

- $A_k, A \in BV([a,b], \mathbb{R}^{n\times n}),\ f_k, f \in G([a,b], \mathbb{R}^n),\ \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N},$ are left-continuous on $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\operatorname{var}_a^b A_k \ :\ k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

WE HAVE: $w_k = (x_k - f_k) - (x - f)$,

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

$$\lim_{k\to\infty} \|h_k\|_\infty = 0, \quad \lim_{k\to\infty} \widetilde{w}_k = 0.$$

By Corollary of the Gronwall Lemma we get

$$\|w_k(t)\|_{\mathbb{R}^n} \le 2\, c_A \left(\|\widetilde{w}_k\|_{\mathbb{R}^n} + 2\|h_k\|_\infty\right) \exp\left(4\, c_A \operatorname{var}_a^t A_k\right) \quad \text{on } [a,b].$$

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Hence $\lim_{k\to\infty} \|w_k\|_\infty = 0$, i.e. $\lim_{n\to\infty} \|x_k - x\|_\infty = 0\,.$ $\square$

Consider

$$x_k' = P_k(t)\,x_k, \quad x_k(a) = \widetilde{x},$$
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where $P_k, P \in L([a,b],\mathcal{L}(\mathbb{R}^n))$ for $k \in \mathbb{N}$.

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Kurzweil & Vorel, 1957

ASSUME:

- there is $m \in L([a,b],\mathbb{R}^1)$ such that $|P_k(t)| \le m(t)$ a.e. on $[a,b]$ for $k \in \mathbb{N}$,
- $\displaystyle\int_a^t P_k\,ds \rightrightarrows \int_a^t P\,ds$ on $[a,b]$.

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$$x_k(t) = \widetilde{x}_k + \int_a^t dA_k\, x_k,$$
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Proposition

ASSUME:

- $\sup\,\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty,$
- $A_k \rightrightarrows A.$

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

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Opial, 1967

ASSUME:

- $\|P_k\|_1 \le p^* < \infty$ pro all $k \in \mathbb{N}$,
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Continuity in weak topology of $L([a,b],\mathbb{R}^n)$

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Zhang & Meng

$P_k \rightharpoonup P$ in $L([a,b],\mathcal{L}(\mathbb{R}^n))$ iff:

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$$\boxed{\text{Opial} \quad \approx \quad \Big[P_k \rightharpoonup P \ \text{ in } L[a,b] \quad \Rightarrow \quad x_k \rightrightarrows x \ \text{ on } [a,b]\Big].}$$

Potentials bounded with the weight

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$$x' = P(t)\,x\,, \qquad x(a) = \widetilde{x}\,.$$

$$\begin{aligned} x_k' &= P_k(t)\,x_k, \qquad x_k(a) = \widetilde{x}\,, \\ x' &= \ P(t)\,x, \qquad x(a) = \widetilde{x}\,. \end{aligned}$$

Opial, 1967

ASSUME:

- $$\lim_{k\to\infty}\left[\left\| \int_a^t P_k\;ds - \int_a^t P\;ds \right\|_\infty \left(1 + \|P_k\|_1\right)\right] = 0.$$

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

Variations bounded with the weight

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \tag{L-k}$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x(s) + f(t) - f(a), \qquad t \in [a,b]. \tag{L}$$

Theorem (Monteiro & M.T.)

ASSUME: $A_k \in BV([a,b],\mathbb{R}^{n\times n})$, $f_k \in G([a,b],\mathbb{R}^n)$, $\widetilde{x}_k \in \mathbb{R}^n$ for $n \in \mathbb{N}$,

- $A \in BV([a,b],\mathbb{R}^{n\times n})$, $f \in BV([a,b],\mathbb{R}^n)$, $\widetilde{x} \in \mathbb{R}^n$,
- $[I - \Delta^- A(t)]^{-1} \in L(X)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b\, A_k\right) \|A_k - A\|_\infty = 0$,
- $\lim_{k\to\infty} \left(1 + \mathrm{var}_a^b\, A_k\right) \|f_k - f\|_\infty = 0$.

THEN: (L) has a unique solution $x \in BV([a,b],\mathbb{R}^{n\times n})$ on $[a,b]$.

MOREOVER: (L-k) has a unique solution x_k for k sufficiently large and $x_k \rightrightarrows x$.

Essential tool for the proof of the previous result is the Kiguradze lemma:

Kiguradze lemma

<u>ASSUME</u>:

- $A,\, A_k \in BV([a,b],\mathbb{R}^{n\times n})$ for $k\in\mathbb{N},,$
- $\det[I-\Delta^- A(t)]\neq 0$ for $t\in(a,b]$,
- $\lim\limits_{k\to\infty}\Big(1+\operatorname{var}_a^b A_k\Big)\,\|A_k-A\|_\infty=0\,.$

<u>THEN</u>: there exist $r^*>0$ and $k_0\in\mathbb{N}$ such that

$$\|x\|_\infty \le r^*\left(|x(a)|+\big(1+\operatorname{var}_a^b A_k\big)\sup_{t\in[a,b]}\left|x(t)-x(a)-\int_a^t d\,A_k x\right|\right)$$

for $x\in G([a,b],\mathbb{R}^n)$ and $k\ge k_0$.

WE ASSUME: for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\|y_n\|_\infty > n\left(\|y_n(a)\|_X + \left(1+\operatorname{var}_a^b A_{k_n}\right)\sup_{t\in[a,b]}\left\|y_n(t)-y_n(a)-\int_a^t d\,[A_{k_n}]\,y_n\right\|_X\right).$$

WE ASSUME: for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\frac{1}{n} > \frac{\|y_n(a)\|_X}{\|y_n\|_\infty} + \left(1+\operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| \frac{y_n(t)}{\|y_n\|_\infty} - \frac{y_n(a)}{\|y_n\|_\infty} - \int_a^t d\,[A_{k_n}]\, \frac{y_n}{\|y_n\|_\infty} \right\|_X$$

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$$\left.\begin{array}{l} \frac{1}{n} > \|u_n(a)\|_X + \left(1+\operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| u_n(t) - u_n(a) - \int_a^t d[A_{k_n}]\, u_n \right\|_X \\ \text{where} \quad u_n(t) = \frac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } t\in[a,b] \text{ and } n\in\mathbb{N}. \end{array}\right\} \implies \|u_n(a)\|_X \to 0.$$

Put $v_n(t) = u_n(t) - u_n(a) - \int_a^t d[A_{k_n}]\, u_n$. Then

$$\|v_n\|_\infty < \frac{1}{n\,(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n\in\mathbb{N} \implies v_n \rightrightarrows 0;$$

$z_n := u_n - v_n \in BV$, $z_n(a) = u_n(a)$, $\|z_n\|_{BV} \le 1 + \operatorname{var}_a^b A_{k_n}$ and

$$z_n(t) = z_n(a) + \int_a^t d[A]\, z_n + h_n(t), \quad h_n(t) = \int_a^t d[A_{k_n} - A]\, z_n + \int_a^t d[A_{k_n}]\, v_n \quad \text{for } t\in[a,b];$$

$$\left.\begin{array}{l} \left\| \int_a^t d[A_{k_n} - A]\, z_n \right\|_X \le 2\, \|A_{k_n} - A\|_\infty\, \|z_n\|_{BV} \le 2\, \|A_{k_n} - A\|_\infty\, (1 + \operatorname{var}_a^b A_{k_n}), \\ \left\| \int_a^t d A_{k_n} v_n \right\|_\infty \le (\operatorname{var}_a^b A_{k_n})\, \|v_n\|_X \le \frac{1}{n}\, \frac{\operatorname{var}_a^b A_{k_n}}{(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \end{array}\right\} \implies \|h_n\|_\infty \to 0.$$

Hence, by the generalized Gronwall inequality

$$\lim_{n\to\infty} \|z_n\|_\infty \le \lim_{n\to\infty} c_A \left(\|z_n(a)\|_X + 2\, \|h_n\|_\infty\right) \exp\left(c_A \operatorname{var}_a^b A\right) = 0.$$

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Sketch of proof of the Opial Type Theorem

$$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

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Sketch of proof of the Opial Type Theorem

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Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0,\infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t \in [a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty\right)\right) \quad \text{for } k \ge k_0.$$

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$$\implies u_k \rightrightarrows 0$$

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$$\implies u_k \rightrightarrows 0 \implies x_k \rightrightarrows x.$$

$$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

$$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X) \text{ on } (a,b] \text{ for } k \ge k_1$$

$$\implies \text{solutions } x_k,\, x \text{ exist for all } k \ge k_0.$$

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a))\,.$$

Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0,\infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty\right)\right) \text{ for } k \ge k_0.$$

$$\implies u_k \rightrightarrows 0 \implies x_k \rightrightarrows x. \qquad \square$$

Remark

Main Theorem could be extended to the case $f \in G([a,b],X)$ if the following convergence assertion was true:

Let $A,\, A_k \in BV([a,b],\mathcal{L}(X))$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty}\Big(1+\operatorname{var}_a^b A_k\Big)\,\|A_k - A\|_\infty = 0.$ Then

$$\int_a^t d\,[A_k]\,f \rightrightarrows \int_a^t d\,[A]\,f \quad \text{for each } f \in G([a,b],X)\,.$$

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However, next example shows that this does not hold.

Let $a=0$, $b=1$, $X=\mathbb{R}$,

$$
\begin{aligned}
n_k &= [k^{3/2}]+1, & \tau_{m,k} &= \frac{1}{2^{n_k-m}} & &\text{if } m\in\{0,1,\dots,n_k\}\,,\\
a_{0,k} &= \frac{2^{n_k}}{k}(-1)^{n_k}, & b_{0,k} &= \frac{1}{k}(-1)^{n_k-1}, & &\\
a_{m,k} &= \frac{2^{n_k-m+1}}{k}(-1)^{n_k-m}, & b_{m,k} &= \frac{3}{k}(-1)^{n_k-m+1} & &\text{if } m\in\{1,2,\dots,n_k-1\}
\end{aligned}
$$

$$
\begin{aligned}
A_k(t) &= \begin{cases} 0 & \text{if } t\in[0,\tau_{0,k}]\,,\\ a_{m,k}\,t+b_{m,k} & \text{if } t\in[\tau_{m,k},\tau_{m+1,k}] \text{ and } m\in\{0,1,\dots,n_k-1\}\,,\end{cases}\\
A(t) &= 0 \quad \text{for } t\in[0,1]\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\operatorname{var}_0^1 A_k \le \frac{1}{k}+\frac{2\,(n_k-1)}{k} \le \frac{1}{k}+2\sqrt{k}<\infty\,,\\
&\Big(1+\operatorname{var}_0^1 A_k\Big)\,\|A_k-A\|_\infty \le \left(1+\frac{2\,n_k-1}{k}\right)\frac{1}{k} \le \frac{1}{k}+\frac{2}{\sqrt{k}}+\frac{1}{k^2}
\end{aligned}
$$

However, if

$$
f(t)=\begin{cases} \dfrac{(-1)^n}{\sqrt[4]{n}} & \text{if } t\in(2^{-n},2^{-(n-1)}] \text{ for some } n\in\mathbb{N},\\ 0 & \text{if } t=0\,,\end{cases} \tag{1}
$$

then f is regulated, $\operatorname{var}_0^1 f=\infty$ and

$$
\int_0^1 d[A_k]\,f \ge \frac{2}{k}\sum_{m=1}^{n_k-1}\frac{1}{\sqrt[4]{m}} > \frac{2}{k}\int_1^{n_k}\frac{1}{\sqrt[4]{t}}\,dt = \frac{8}{3\,k}\Big(\sqrt[4]{(n_k)^3}-1\Big), \tag{2}
$$

where the right hand side tends to ∞ for $k\to\infty$.

Example

$$x_k(t) = \widetilde{x} + \int_0^t dA_k\, x_k, \quad t \in [0,1],$$

where

$$A_k(t) = P\,t + I \left\{ \begin{array}{ll} k\,t & \text{if } 0 \le t \le 1/k, \\ 1 & \text{if } \frac{1}{k} \le t \le 1 \end{array} \right\}$$

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On the other hand, we have $A_k \rightharpoonup^* A$ in $NBV[a,b] = (C[a,b])^*$ and

$$x_k(t) = \left\{ \begin{array}{ll} exp(P\,t + k\,I\,t)\widetilde{x} & \text{if } \ 0 < t \le 1/k, \\ \exp(P\,t + I)\widetilde{x} & \text{if } \ 1/k \le t \le 1 \end{array} \right\} \to x_0(t) = \left\{ \begin{array}{ll} \widetilde{x} & \text{if } \ t = 0, \\ \exp(P\,t + I)\widetilde{x} & \text{if } \ 0 < t \le 1 \end{array} \right\} \text{ on } [0,1].$$

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$\boxed{\text{analog of Opial \& Zhang \& Meng result is not true pro GLDE's}}$

$$x_k(t) = \widetilde{x}_k + \int_a^t d A_k\, x + f_k(t) - f_k(a), \quad t \in [a,b]\,,$$

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Halas

LET:

- $\sup\,\left\{\mathrm{var}_a^b\,A_k : k \in \mathbb{N}\right\} < \infty,$
- $A_k \rightrightarrows A,\quad f_k \rightrightarrows f$ locally on $(a,b]$ and $\widetilde{x}_k \to \widetilde{x},$
- $\forall\,\varepsilon > 0\ \exists\,\delta > 0$ such that $\forall\,t \in (a, a+\delta)\ \exists\,k_0 \in \mathbb{N}$ such that

  $|x_k(a) - \widetilde{x} - \Delta^+ A(a)\,\widetilde{x} - \Delta^+ f(a)| < \varepsilon$ for all $k \geq k_0$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d\,A_k\,x + f_k(t) - f_k(a), \quad t \in [a,b]\,,$$
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THEN: $x_k \to x$ on $[a,b]$, while $x_k \rightrightarrows x$ locally on $(a,b]$.

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LEMMA applies to the last EXAMPLE with

$$A(t) = P\,t + I \quad \text{and} \quad f(t) = (\widetilde{y} - \widetilde{x})\,\chi_{(0,1]}(t), \quad \text{where } \widetilde{y} = \exp(I)\,\widetilde{x}.$$

Assume:

$$\det[I - \Delta^{-}A(t)] \neq 0 \;\; \textit{and} \;\; \det[I + \Delta^{+}A(s)] \neq 0 \quad \text{for} \;\; t \in (a, b\,],\, s \in [a, b)\,.$$

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Theorem

There is uniquely determined matrix valued function $U : [a, b] \times [a, b] \to \mathbb{R}^{n \times n}$ such that

$$U(t, s) = I + \int_s^t d\left[A(\tau)\right] U(\tau, s) \quad \text{for} \ \ t, s \in [a, b].$$

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- $U(\cdot,s) \in BV([a,b],\mathbb{R}^{n\times n})$ for every $s \in [a,b]$,
- $U(t,t) = I$ for every $t \in [a,b]$,
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Corollary

Let: $t_0 \in [a, b]$ and $\widetilde{x} \in X$. Then: $x : [a, b] \to X$ is a solution of

$$x(t) - \widetilde{x} - \int_{t_0}^t d\, A\, x = 0 \quad \text{on} \;\; [a, b]$$

iff $x(t) = U(t, t_0)\, \widetilde{x}$ for $t \in [a, b]$.

Variation-of-constants formula

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d A\, x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

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<u>ASSUME</u>: $t_0 \in [a,b]$, $A \in BV([a,b],\mathbb{R}^{n\times n})$,

$$\det\,[I - \Delta^{-}A(t)] \neq 0 \quad \text{and} \quad \det\,[I + \Delta^{+}A(s)] \neq 0 \quad \text{for } t \in (a,b\,],\ s \in [a,b)$$

and U is the Cauchy matrix function for (L).

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} d\,A\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

Theorem

<u>ASSUME</u>: $t_0 \in [a,b]$, $A \in BV([a,b],\mathbb{R}^{n\times n})$,

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and U is the Cauchy matrix function for (L).

<u>THEN</u>: (L) has for every $\widetilde{x} \in \mathbb{R}^n$ and $f \in G([a,b],R^n)$ a unique solution x on $[a,b]$.

This solution is given by

$$x(t) = U(t,t_0)\,\widetilde{x} + f(t) - f(t_0) - \int_{t_0}^{t} d_s[U(t,s)\,]\,\big(f(s) - f(t_0)\big) \quad \text{for } t \in [a,b]\,.$$

8. MEASURE EQUATIONS

Let

$$A(t) = \begin{pmatrix} 0 & P(t) \\ Q(t) & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

where $P, Q \in BV([a,b], \mathbb{R}^{n\times n})$, $g, h \in BV([a,b], \mathbb{R}^n)$ and $\widetilde{y}, \widetilde{z} \in \mathbb{R}^n$.

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Then

$$x(t) = \widetilde{x} + \int_a^t d A\, x + f(t) - f(a)$$

reduces to

$$y(t) = \widetilde{y} + \int_a^t d P\, z + g(t) - g(a),$$

$$z(t) = \widetilde{z} + \int_a^t d Q\, y + h(t) - h(a)$$

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and $\quad \det[I - \Delta^- A(t)] \neq 0 \quad$ iff

$$\det[I - \Delta^- Q(t)\, \Delta^- P(t)] \neq 0 \text{for } \ t \in (a,b]$$

or

$$\det[I - \Delta^- P(t)\, \Delta^- Q(t)] \neq 0 \text{for } \ t \in (a,b]$$

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t dP_k\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t dQ_k\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \tag{S-k}$$

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Second order measure equations

Consider systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t d P_k\, z_k + g_k(t) - g_k(a), \\ z_k(t) &= \widetilde{z}_k + \int_a^t d Q_k\, y_k + h_k(t) - h_k(a), \end{aligned}\right\} \qquad \text{(S-k)}$$

$$\left.\begin{aligned} y(t) &= \widetilde{y} + \int_a^t d P\, z + g(t) - g(a), \\ z(t) &= \widetilde{z} + \int_a^t d Q\, y + h(t) - h(a). \end{aligned}\right\} \qquad \text{(S)}$$

Corollary

<u>ASSUME</u>: $P, Q \in BV([a,b], \mathbb{R}^{n\times n}),\ g, h \in BV([a,b], \mathbb{R}^n),\ \widetilde{y}, \widetilde{z} \in \mathbb{R}^n$,

- $\det[I - \Delta^- Q(t)\, \Delta^- P(t)] \neq 0$ or $\det[I - \Delta^- P(t)\, \Delta^- Q(t)] \neq 0$ for $t \in (a,b])$,
- $\lim\limits_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0, \qquad \lim\limits_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0,$
- $\lim\limits_{k\to\infty} \big(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\big)\big(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\big) = 0,$
- $\lim\limits_{k\to\infty} \big(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\big)\big(\|g_k - g\|_\infty + \|h_k - h\|_\infty\big) = 0.$

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- $\lim_{k\to\infty} \left(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\right)\left(\|g_k - g\|_\infty + \|h_k - h\|_\infty\right) = 0.$

THEN:

- (S) has a unique solution $(y, z) \in BV([a,b], \mathbb{R}^n) \times BV([a,b], \mathbb{R}^n)$ on $[a,b]$,
- (S-k) has a unique solution $(y_k, z_k) \in G([a,b], \mathbb{R}^n) \times G([a,b], \mathbb{R}^n))$ on $[a,b]$ for k sufficiently large,
- $\lim_{k\to\infty} \|y_k - y\|_\infty + \|z_k - z\|_\infty = 0.$

Meng and Zhang:

$$d\,y^{\bullet} + d\,[\mu_k(t)]\,y = 0, \quad y(0) = \widetilde{y},\, y^{\bullet}(0) = \widetilde{z},\; k \in \mathbb{N}\,, \tag{mz-k}$$

where $\mu_k \in BV[a,b]$ are right-continuous, $\widetilde{y},\, \widetilde{z} \in \mathbb{R}$ and $y^{\bullet}$ is the generalized right-derivative of y.

They proved that the weak* convergence $\mu_k \to \mu$ yields

$$y_k \rightrightarrows y,\; y_k^{\bullet} \to y^{\bullet} \text{ in weak* topology and } y_k^{\bullet}(1) \to y^{\bullet}(1).$$

(S-k) reduce to (mz-k) when

$$n=1,\; [a,b]=[0,1],\; P_k(t)=t,\; Q_k(t)=\mu_k(t) \text{ and } g_k, h_k \text{ are constant.}$$

Similarly, (S) reduces to

$$d\,y^{\bullet} + d\,[\mu(t)]\,y = 0, \quad y(0) = \widetilde{y},\, y^{\bullet}(0) = \widetilde{z} \tag{mz}$$

if

$$P(t)=t,\; Q(t)=\mu(t) \text{ and } g, h \text{ are constant.}$$

As existence conditions are obviously satisfied, by our **Corollary** we have

$$\lim_{k\to\infty}\left(\|y_k - y\|_\infty + \|y_k^{\bullet} - y^{\bullet}\|_\infty\right) = 0 \quad \text{whenever} \quad \lim_{k\to\infty}\left(1 + \operatorname{var}_0^1 \mu_k\right)\|\mu_k - \mu\|_\infty = 0.$$

9. TIME SCALES

Time scales: nonempty and closed subset $\mathbb{T}$ of $\mathbb{R}$.

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For $a, b \in \mathbb{T}$, we set $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$.

$$\sigma(t) := \inf \big((t,b] \cap \mathbb{T}\big) \quad \text{is the } \textbf{forward jump operator},$$

$$\rho(t) := \sup \big([a,t) \cap \mathbb{T}\big) \quad \text{is the } \textbf{backward jump operator}$$

and

$$\mu(t) = \sigma(t) - t \quad \text{is the } \textbf{graininess} \text{ of the time scale.}$$

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$\mu(t) = \sigma(t) - t$ is the **graininess** of the time scale.

For a given $\delta > 0$, a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\} \subset [a,b]_{\mathbb{T}}$ of $[a,b]$ is said to be δ-**fine** if

either $\alpha_i - \alpha_{i-1} < \delta$ or $\rho(\alpha_i) = \alpha_{i-1}$.

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$$\text{either} \quad \alpha_i - \alpha_{i-1} < \delta \quad \text{or} \quad \rho(\alpha_i) = \alpha_{i-1}.$$

We also say that $P = (D, \xi)$ is a **tagged division** of $[a,b]_{\mathbb{T}}$ if

$$\xi = \{\xi_1, \dots \xi_{\nu(D)}\} \quad \text{and} \quad \xi_i \in [\alpha_{i-1}, \alpha_i) \cap \mathbb{T} \quad \text{for } i \in \{1, \dots, \nu(D)\}.$$

Then

$$I = \int_a^b f(t)\,\Delta t$$

iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\Big|\sum_{i=1}^{\nu(D)} f(\xi_i)(\alpha_i - \alpha_{i-1}) - I\Big| < \varepsilon \quad \text{for all } \delta\text{−fine tagged divisions } P = (D, \xi) \text{ of } [a,b]_{\mathbb{T}}.$$

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Linear dynamical equations on time scales

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Proposition (Slavík)

ASSUME: $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is rd-continuous,

$$F_1(t) = \int_a^t f(s)\,\Delta s \quad \text{and} \quad F_2(t) = \int_a^t f(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t \in [a,b].$$

THEN: $F_2 = F_1 \circ \widetilde{\sigma}$.

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THEN: $F_2=F_1\circ\widetilde{\sigma}$.

Consider equation

$$y(t)=\widetilde{y}+\int_a^t \big[P(s)\,y(s)+h(s)\big]\,\Delta s\,,\qquad t\in[a,b]_{\mathbb{T}}\,, \tag{D}$$

where $P:[a,b]_{\mathbb{T}}\to\mathcal{L}(\mathbb{R}^n)$ and $h:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$ are rd-continuous on $[a,b]_{\mathbb{T}}$, and put

Linear dynamical equations on time scales

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Consider equation

$$y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{D}$$

where $P : [a,b]_{\mathbb{T}} \to \mathcal{L}(\mathbb{R}^n)$ and $h : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ are rd-continuous on $[a,b]_{\mathbb{T}}$, and put

$$A(t) = \int_a^t P(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{a} \quad f(t) = \int_a^t h(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t \in [a,b]\,.$$

Linear dynamical equations on time scales

Put $\widetilde{\sigma}(t) := \inf\big([t,b]\cap\mathbb{T}\big)$ (recall: $\sigma(t) := \inf\big((t,b]\cap\mathbb{T}\big)$).

Proposition (Slavík)

ASSUME: $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is rd-continuous,

$$F_1(t) = \int_a^t f(s)\,\Delta s \quad \text{and} \quad F_2(t) = \int_a^t f(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t\in[a,b].$$

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$$y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t\in[a,b]_{\mathbb{T}}\,, \tag{D}$$

where $P : [a,b]_{\mathbb{T}} \to \mathcal{L}(\mathbb{R}^n)$ and $h : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ are rd-continuous on $[a,b]_{\mathbb{T}}$, and put

$$A(t) = \int_a^t P(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{a} \quad f(t) = \int_a^t h(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t\in[a,b]\,.$$

Theorem (Slavík)

- If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (LD), then $x = y\circ\widetilde{\sigma}$ is a solution of

$$x(t) = \widetilde{y} + \int_a^t d\,A\,x + f(t) - f(a)\,, \quad t\in[a,b]\,. \tag{L}$$

- If x is a solution of (GL) and $y = x|_{\mathbb{T}}$, then y is a solution of (LD).

$$y(t) = \tilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{LD}$$

$$y(t) = \tilde{y}_k + \int_a^t \big[P_k(s)\,y(s) + h_k(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,, \tag{LD-k}$$

Corollary

ASSUME: P, $P_k\colon [a,b]_{\mathbb{T}} \to \mathcal{L}(\mathbb{R}^n)$, h, $h_k\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ for $k \in \mathbb{N}$ are rd-continuous in $[a,b]_{\mathbb{T}}$,

$$\alpha_k = \sup_{t\in[a,b]_{\mathbb{T}}} \|P_k(t)\|_{L(\mathbb{R}^n)} + \sup_{t\in[a,b]_{\mathbb{T}}} \|h_k(t)\|_{\mathbb{R}^n} \quad \text{for } k \in \mathbb{N}\,,$$

$$\lim_{k\to\infty} \|\tilde{y}_k - \tilde{y}\|_{\mathbb{R}^n} = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_k(s) - P(s))\,\Delta s \Big\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (h_k(s) - h(s))\,\Delta s \Big\|_{L(\mathbb{R}^n)} \big[1 + \alpha_k\big] = 0\,.$$

THEN: (LD) has a solution y, (LD-k) has a solution y_k for $k \in \mathbb{N}$ sufficiently large and

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \|y_k(t) - y(t)\|_{\mathbb{R}^n} = 0\,.$$

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